Knot Determinants

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Abstract

Knot invariants offer a way of discriminating between knots using algebraic values that do not change with respect to ambient isotopy. This paper considers the determinant, which is one of the classic knot invariants arising from the coloring matrix related to the knot diagram. Starting from the basic concepts of Fox colorings, the coloring matrices are introduced and the concept of determinant is defined as the absolute value of any minor of the coloring matrix. It is demonstrated that the determinant is independent of the minor chosen, strand labelling and knot projection, thus making it a valid knot invariant. The multiplication rule of the determinants regarding the connected sum of two knots is proved by the application of elementary row and column transformations to construct the block-diagonal coloring matrix. Torus knots are discussed in the last section, where the determinant of such knots is characterized via their parameters' parity.

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Introduction

Knot theory is a branch of topology which investigates closed curves in three-dimensional spaces (Osorio & Olhava, 2022). Unlike knots encountered in everyday life, knots in mathematics have the ends of strands connected to form a closed loop.

To distinguish between knots, mathematicians use knot invariants, properties that remain unchanged even under continuous deformation (Colin Conrad Adams, 1994). Examples of these include tricolorability, knot polynomials, and knot determinants. In particular, the knot invariant studied in this paper is the knot determinant. This invariant comes from Fox colorings, which assign specific values to various strands of a knot diagram through a system of linear equations (Livingston, 1993). The determinant obtained from this system contains important information about a knot, including its colorability properties. Specifically, a knot is p-colorable if and only if its determinant is divisible by p, where $p$is an odd prime (Livingston, 1993).

Previous work has shown that **determinants** are multiplicative under connected sums: if knots K and L are combined, the det(K#L) = det(K) x det(L) (Dixon, 2011). This will later be explored in further detail.

The purpose of this paper is to investigate the knot determinant and its behavior under connected sums. After reviewing the necessary background on knot theory and Fox colorings, we derive and extend the multiplicative determinant formula for composite knots. Finally, we apply these results to the torus knot famil**y** and discuss their implications for knot colorability.

Background

To study knot determinants, we require several fundamental concepts from knot theory.
Definition 2.1 A knot is a simple closed curve embedded in three-dimensional space (Colin Conrad Adams, 1994). Since knots exist in three dimensions, they are often represented using two-dimensional knot diagrams. In these diagrams, crossings show where one strand passes over another. The under-strand is broken to indicate the crossing relationship. For example, see Figure 3 of the trefoil knot.

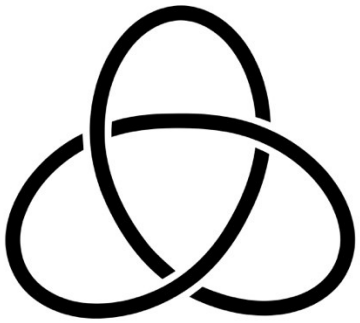

Figure 3. A 2-D projection of the trefoil knot

(Wikipedia Contributors, 2007)

Two knot diagrams may appear different while representing the same knot.

Definition 2.2. Two knots K and J are equivalent if one can be continuously deformed into the other without cutting the strand or allowing it to pass through itself (Calegari, 2025). This deformation is known as an ambient isotopy.

Many of these knot deformations can be described in the two-dimensional plane of a knot diagram. These are known as planar isotopies as they do not alter the crossing structure of the knot. This may be shrinking, bending, or sliding of strands. Another type of knot deformation is known as the Reidemeister moves, which fundamentally change the configuration of crossings in a knot in the three-dimensional space (Calegari, 2025). There are exactly three Reidemeister moves.

Definition 2.3. A Reidemeister move is one of the following transformations of a knot diagram (Calegari, 2025):

- Type I: introduces or removes a twist
- Type II: introduces or removes a pair of crossings
- Type III: slides one strand across a crossing formed by two other strands.

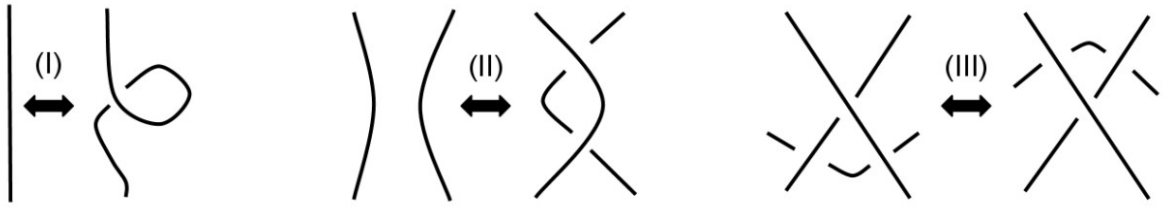

Figure 4. The three Reidemeister moves

(Henrich & Kauffman, 2026)

The study of knot equivalence thus motivates the search for knot invariants.

Definition 2.4. A knot invariant is a quantity of property that remains unchanged under ambient isotopy (Calegari, 2025). Examples include crossing numbers, tricolorability, knot polynomials, and the knot determinant.

Though knots themselves are topological structures, many of their characteristics may be analyzed using linear algebra. For instance, by applying linear algebra methods, a relationship can be established between a knot diagram and a set of linear equations by means of Fox coloring. Such equations can be written in terms of matrices, making it possible to analyze knot properties (Calegari, 2025).

A matrix refers to a rectangular array of numbers arranged in rows and columns. Matrices are commonly used to express systems of linear equations in which each element of the matrix represents a coefficient in the system. Regarding Fox coloring, the relations in colors for every crossing of a knot may be represented by a matrix called the coloring matrix (Calegari, 2025).

An interesting feature concerning matrices is the determinant of a square matrix. This is a number obtained from the elements of the matrix and provides us with information concerning the system of equations. The determinant of a knot is determined by a minor of the coloring matrix and is considered a knot invariant (Calegari, 2025).

Elementary row operations can be used to simplify the study of coloring matrices. There are three types of row operations: swapping two rows, multiplying a row by some number not equal to zero, and adding a product of one row to another (University of California Davis, n.d.). Such operations are useful because they allow matrices to be transformed into simpler forms while preserving important information about the system they represent. Throughout this paper, row operations will be used to analyze coloring matrices and establish properties of knot determinants.

Fox Colorings and Coloring Matrices

Tricolorability is an example of a knot invariant.

Definition 3.1. A knot diagram is said to be tricolorable if each strand can be assigned one of three colors (Shepherd & Schultens, 2022). At every crossing, either all three colors are present or only one color appears. At least two colors must be used in the coloring. See the below figure of the tricoloring of the trefoil knot.

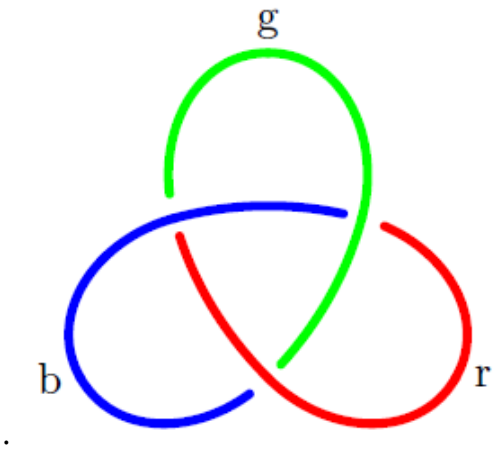

Figure 5. Tri-colorability of a trefoil

(Moskovich, 2014)

To prove that tricolorability is a knot invariant, we can show that tricolorability is maintained through each of the Reidemeister moves.

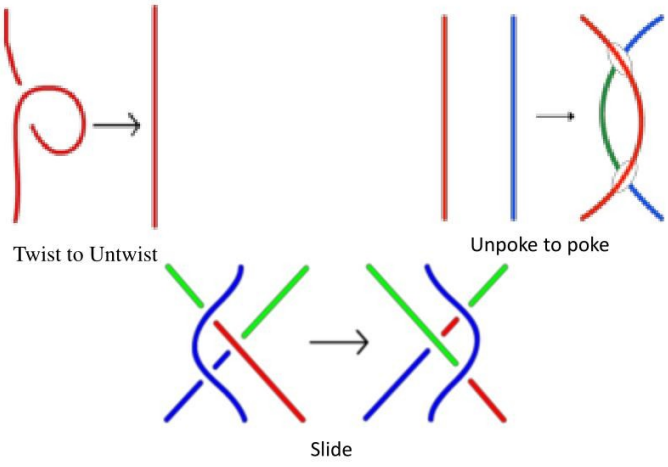

Figure 6. Tricolorability of Reidemeister moves

(Dunderdale, 2014)

Since each of the Reidemeister moves can be made without affecting its tricolorability, they are a knot invariant.

Tricolorability distinguishes certain knots, such as the trefoil and unknot, but cannot distinguish knots that are both tricolorable or both non-tricolorable. Consequently, a more powerful invariant is required: the Fox colorings, which extend the notion of tricolorability by allowing colors chosen from an arbitrary finite set of integers modulo (r).

Definition 3.2. Let r be a positive integer. A Fox p-coloring of a knot diagram is an assignment of an element of Z(mod p) to each strand such that at every crossing,

$$2x = y + z \pmod{p}$$

Where x is the value assigned to the overstrand, and y and z are values assigned to the two understrands (Livingston, 1993).

A knot that admits a nontrivial Fox r-coloring is said to be p-colorable. Note that when $p = 3$, Fox colorings are reduced to tricolorings, meaning that tricolorability is one specific case of Fox colorability (Livingston, 1993).

The coloring conditions at each crossing can be used to produce a system of linear equations.

Definition 3.3. Let K be a knot with n crossings. The coloring matrix of K, M_K, is then the $n \times n$ matrix whose rows denote crossings and whose columns denote strands. For each crossing, the corresponding row contains (Livingston, 1993):

- A 2 in the column representing the overstrand
- A -1 in the columns representing the two understrands,
- And a 0 in all remaining columns.

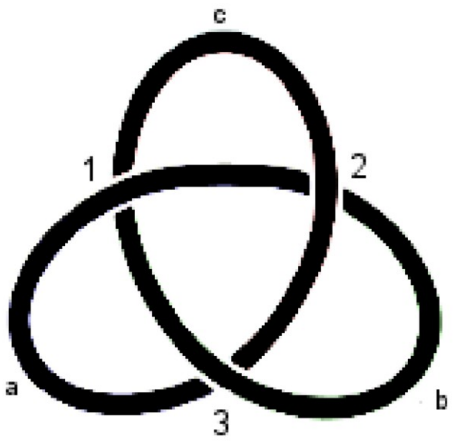

Figure 7. Labelled trefoil knot

(Dixon, 2011)

For example, the coloring matrix of the trefoil knot in Figure 7 is

$$M=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

In each row, the sum of the entries is zero. This means that the determinant of the entire coloring matrix is always equal to zero. In other words, the rows of the coloring matrix are linearly dependent and cannot be used directly as a knot invariant.

To extract a meaningful invariant from this matrix, we can instead examine smaller matrices within by removing any row and column from the coloring matrix.

Definition 3.4. Let M_K be the coloring matrix of a knot *K*. Then, a minor of M_K is the determinant of the resulting matrix after any one row and column are removed from M_K.

For example, removing the third row and third column from the trefoil coloring matrix results in the following minor matrix.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The determinant of this matrix is 3.

It can be calculated that all the minors of a coloring matrix have the same absolute value. This means that an invariant can be defined so that it is associated with the knot itself rather than a particular choice of row and column.

Definition 3.5. The determinant of a knot K, denoted det(K), is the absolute value of any minor of its coloring matrix.

This determinant lends substantially more information than tricolorability alone. It can distinguish between different knots and also provides the existence of Fox colorings. In the following section, we will establish the relationship between knot determinants and colorability and investigate how determinants behave under connected sums of knots.

Theorem 3.6. The determinant of a knot does not depend on the choice of minor, the labelling convention of the knot, or the projection chosen. As a result, the determinant is a knot invariant.
Lemma 3.7. If a row and column are added to a square matrix such that all entries are zero except for a single 1 on the diagonal, then the determinant remains unchanged.

Proof. Let A be an $n \times n$ matrix and let B be the $(n+1) \times (n+1)$ matrix formed by the addition of a new row and column containing zeros, with the exception of a single 1 on the diagonal.

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

Expanding det(B) along the last row (or column), every term is zero except the diagonal entry 1. Therefore,

$$det(B) = 1 \times det(A) = det(A).$$

And, adding such a row or column as shown in B results in a determinant equal to A.
Lemma 3.8. Suppose A is an $n \times n$ matrix whose rows and columns each sum to zero. Then, every minor of A has the same absolute value.

Proof. Let A be an $n \times n$ matrix such that every column sums to zero. Let B be the $n \times n$ matrix whose entries are all equal to 1. Then consider the matrix $C = A + B$.

$$A=(a_{ij})_{n\times n}$$

$$B=\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}$$

We must show that the determinant of such a matrix *C* can be written in terms of any minor of A. Since det (A+B) is fixed, this will show that all minors of A must have the same absolute value.

Choosing any row i and any column j, we want to relate det(A+B) to the minor formed by deleting row i and column j from A.

We first perform the row operation

$$R_i \leftarrow R_1+R_2+\cdots R_n,$$

which replaces the i-th row by the sum of all rows of C. Adding multiples of other rows to a row does not change the determinant.

Now consider the *k*-th column of the *i*-th row. The sum of the entries in column *k* are

$$S=(a_{1k}+1)+(a_{2k}+1)+\cdots+(a_{nk}+1),$$

Alternatively,

$$S=a_{1k}+a_{2k}+\cdots+a_{nk}+(1+1+\cdots+1).$$

Since every column of A sums to zero,

$$S=1+1+\cdots+1=n.$$

This argument holds for every column *k*, thus

$$R_i=[n,n,\cdots,n].$$

Next, add all other columns in C to column *j*. Using the same reasoning, the entries from A cancel in column *j*. The (*i*,*j*) entry becomes n^2, shown in brackets, and every other entry in column *j* becomes *n*.

$$\begin{bmatrix} ¿ & ¿ & \cdots & n & ¿ \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ ¿ & ¿ & \ddots & n & ¿ \\ n & n & \cdots & (n^2) & n \\ ¿ & ¿ & \cdots & n & ¿ \end{bmatrix}$$

Now, we can factor out n from row i and subtract row i from every other row. This makes every entry in column j, apart from the (i,j) entry, equal to zero. Therefore, the remaining matrix has the form

$$\begin{bmatrix} ¿ & ¿ & 0 & \cdots & ¿ \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ ¿ & ¿ & 0 & \cdots & ¿ \\ ¿ & ¿ & n & \cdots & ¿ \\ ¿ & ¿ & 0 & \cdots & ¿ \end{bmatrix}$$

where expanding along column j results in

$$det(A+B) = \pm n^2 M_{ij},$$

Where M_{ij} is the minor of A obtained by deleting row i and column j. Here, the stars denotes the corresponding entry of the original matrix A, unchanged by the row and column operations. The only modified entries are those in row i and column j.

Since the selection of row i and column j were arbitrary, this same process holds for any minor. Consequently, any two minors M_{ij} and M_{kl} satisfy

$$\pm n^2 M_{ij} = \pm n^2 M_{kl}.$$

Then, if we divide both sides by n^2, we obtain

$$\pm M_{ij} = \pm M_{kl}.$$

Hence, all minors of A are equal up to sign and thus have the same absolute value.

$$|M_{ij}| = |M_{kl}|.$$

Lemma 3.9. Next, we must prove the determinant of a knot is independent of the labelling convention of the knot.

Proof. A coloring matrix is dependent upon how we label strands and crossings of a knot diagram. Relabeling the strands corresponds to permuting the columns of a coloring matrix, while relabeling the crossings corresponds to permuting the rows.

From linear algebra, interchanging any two rows multiplies the determinant by -1. Interchanging any two columns also multiplies the determinant by -1. Since relabeling is achieved through the rearrangement of rows and columns, it may change the sign of a minor but has no effect on its absolute value. Therefore, the determinant of a knot is independent of its labelling convention.

For the theorem to hold, it remains to show the determinant to be independent of the knot projection chosen. Since any two projections of the same knot are related by a series of Reidemeister moves, we can show that the determinant is preserved under each of the Reidemeister moves.

Type I. This move introduces or removes a single twist in the knot diagram. At the new crossing, one strand acts as both the overstrand and the understrand. Hence, the corresponding Fox coloring equation simplifies to an existing relationship, and does not add any additional constraints on the coloring matrix. Consequently, the coloring matrix gains redundant information that does not alter existing minors. Therefore, the determinant is preserved under Type I moves.

Type II. This move introduces or removes two crossings. Suppose this move introduces two new arc variables, b and c. This introduces equations

$$2a-b-c=0$$

and

$$2c-a-b=0$$

This introduces the submatrix

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then, we can perform the following row operations:

$$R_2 \leftarrow R_1 + R_2$$

$$R_2 \leftarrow 2R_2 - R_1$$

$$R_2 \leftarrow \frac{1}{3}R_2$$

$$R_1 \leftarrow R_1 + R_2$$

$$R_1 \leftarrow \frac{1}{2}R_1$$

The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

This means that

$$a = b = c.$$

Therefore, the variables b and c are redundant and can be eliminated from the system as they contribute only trivial pivot entries, or 1's on the diagonal. This means the enlarged matrix is row-equivalent to

$$M' = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

Where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can apply the block determinant rule:

$$det(M') = det(A)\,det(I),$$

And since $det(I) = 1$,

We get

$$det(M') = det(A).$$

Type III. This move slides one strand over or under a crossing. Unlike Type I and II moves, this move does not alter the number of crossings or strands; it merely shifts the local arrangement of the crossings. The coloring equations before and after the Reidemeister III move are therefore equivalent, so the absolute value of the minors does not change. By definition, the determinant is also preserved under Type III moves.

Determinants of Composite knots

Composite knots are the product of two connected knots.

Definition 4.1. Let K and L be knots, the connected sum, also known as a composite knot, of K and L, denoted K#L, is the knot produced by removing a small arc from each knot and joining the resulting endpairs so that no additional crossings are introduced (Colin Conrad Adams, 1994).

However, not every knot can be expressed as the connected sum of two nontrivial knots. If knots can be broken down into further, smaller knots, they are known as a composite knot, and if they are unable to be further decomposed, they are known as a prime knot.

Definition 4.2. A knot K is composite only if there exist nontrivial knots K_1 and K_2 such that

$$K = K_1 ¿ K_2.$$

This connected sum operation allows us to create more complex knots from simpler ones (Roberts, 2015). This raises the question of how knot invariants behave under this operation. Specifically: what is the relationship between the determinant of a composite knot, and the determinant of its parts?

As an illustrative example, consider the connected sum of a trefoil and a figure-eight.

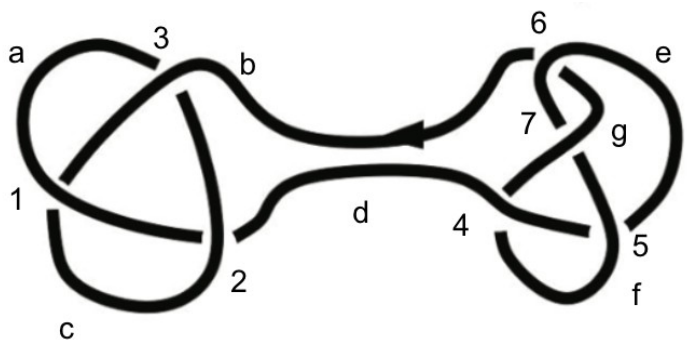

Figure 8. Connected sum of a trefoil and a figure-eight
(Michigan State University, 2024)

Let M_K be the coloring matrix of the trefoil and M_L be the coloring matrix of the figure-eight where,

$$M_K = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} M_L = \begin{bmatrix} 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Using the labelling convention in the figure above, the coloring matrix of the composite knot is

$$M_{K¿}L¿ = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

To compute the determinant, we can remove the fourth row and fourth column, which correspond to the connecting strand. This gives

$$M' = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

There are a few things that can be observed:

- The upper-left 3x3 block closely resembles the trefoil coloring matrix
- The lower-right 3x3 block closely resembles a first-minor of the figure-eight coloring matrix, except for the entry -1 in the fifth row, second column.

We can use the row operation,

$$R_5 \leftarrow R_5 + R_1 + R_2 + R_3,$$

Which eliminates the off-diagonal entry, yielding a block diagonal matrix where

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

The matrix is now block diagonal, and

$$det(M') = det\begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} det\begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The following theorem shows this property holds in general.

Theorem 4.3. For any knots K and L,

$$det(K¿L) = det(K)\,det\,(L)$$

Proof. Let $M_k = (a_{ij})$ and $M_L = (b_{ij})$ be the coloring matrices associated with knots K and L, respectively. Suppose M_k is an $n \times n$ matrix and M_L is an $m \times m$ matrix.

To create the composite knot, we choose one strand of K and one strand of L to form the connected sum. After relabeling rows and columns as necessary, we can assume the strand of K being joined corresponds to column *n*, and the strand of L corresponds to column *n+1*.
The coloring matrix of K#L can be arranged to be almost block diagonal, having the form

$$M = \begin{bmatrix} M_K & 0 \\ 0 & M_L \end{bmatrix},$$

Except for the entries caused by the two connecting strands—the only places where the two knot matrices interact.

To compute the determinant of K#L, we can remove one row and one column to obtain a minor. It has already been proven that the determinant of a knot is independent of the choice of minor; thus we may choose the row and column corresponding to one of the connecting strands.

The resulting minor has the form

$$M=\begin{bmatrix} M_K^{''} & 0 \\ E & M_L^{'} \end{bmatrix}.$$

Here, $M_L^{'}$ is a minor of M_L and the matrix $M_K^{''}$ is the same as M_K apart from one entry, originally (-1), which has become 0. If this changed entry is in row *r* and column *s*,

$$M_K^{''}=M_K \, except \, a_{rs}=-1 \, is \, replaced \, by \, 0$$

The block E contains the remaining connection between the two knots. Since only one strand is shared, E has only one nonzero entry, a -1 in column *s*. Schematically, this looks like

$$M=\begin{bmatrix} M_K^{''} & 0 \\ 0\cdots -1\cdots 0 & M_L^{'} \end{bmatrix},$$

We can attempt to eliminate the -1 entry by row operations.

Since every column of a coloring matrix sums to zero, the columns of M_K satisfy

$$\sum_{i=1}^{n} a_{ij}=0$$

For every column *j*. However, in $M_K^{''}$, the entry a_{rs} is replaced by 0, meaning the column s now has column sum 1, while every other column still has column sum 0.

Now if we apply the row operation

$$R \leftarrow R+R_1+R_2+\cdots+R_n$$

To the row containing the off-diagonal -1, that entry becomes

$$-1+1=0.$$

This means the matrix is now block diagonal:

$$M=\begin{bmatrix} M_K^{''} & 0 \\ 0 & M_L^{'} \end{bmatrix}.$$

Therefore,

$$det(M)=det\begin{bmatrix} M_K^{''} & 0 \\ 0 & M_L^{'} \end{bmatrix},$$

And

$$det(M)=det\left(M_K^{''}\right)det\left(M_L^{'}\right).$$

Since $M_L^{'}$ is a minor of M_L,

$$\left|det\left(M_L^{'}\right)\right| = det\left(L\right).$$

We must still show that

$$\left|det\left(M_K^{''}\right)\right| = det\left(K\right).$$

If we add all rows of $M_K^{''}$ except row *r* to row *r*, row *r* now has a single nonzero entry, 1, in column *s*. If we expand along the row with a single 1. $M_K^{''}$ becomes

$$\begin{bmatrix} M_K^{'} \\ 0 \cdots 1 \cdots 0 \end{bmatrix},$$

Where deleting row *r* and column *s* gives a minor of M_K.

If we then expand along the row with the single 1, we conclude

$$det\left(M_K^{''}\right) = \pm det\left(M_K^{'}\right) = det\left(K\right).$$

Hence,

$$det\left(K ¿ L\right) = det\left(K\right) det\left(L\right).$$

Now, we can prove by induction that this lemma holds true for any number of knots, *n*, composed together.

Corrollary 4.4. Let

$$K = K_1 ¿ K_1 ¿ \cdots K_n.$$

Then

$$det\left(K\right) = \prod_{i=1}^{n} det\left(K_i\right).$$

Where $n \geq 2, n \in Z^{+¿.¿}$

Proof. We prove by induction on *n*. For *n=2*, the result is exactly Theorem 4.3.

Assume the result holds for *n=k*. Namely,

$$det\left(K_1 ¿ K_1 ¿ \cdots K_k\right) = \prod_{i=1}^{k} det\left(K_i\right).$$

Then, consider the connected sum of *k+1* knots:

$$K = K_1 ¿ K_1 ¿ \cdots K_k ¿ K_{k+1}.$$

We can treat the first *k* knots as one knot:

$$det\left(\left(K_1 ¿ K_1 ¿ \cdots K_k\right) ¿ K_{k+1}\right) = det\left(K_1 ¿ K_1 ¿ \cdots K_k\right) \times det\left(K_{k+1}\right).$$

By the induction hypothesis,

$$det\left(K_1 ¿ K_1 ¿ \cdots K_k\right) = \prod_{i=1}^{k} det\left(K_i\right).$$

Hence,

$$det\ \left(K_1 ¿ K_1 ¿ \cdots K_k ¿ K_{k+1}\right) = \prod_{i=1}^{k} det\left(K_i\right) \times det\left(K_{k+1}\right) = ¿ \prod_{i=1}^{k+1} det\left(K_i\right). ¿$$

Thus, the determinant of any connected sum is equal to the product of the determinants of its constituent knots.

Application of knot determinants

The determinant is a particularly useful tool when applied to entire families of knots. Two important families of knots are torus knots and twist knots. These families contain an infinite number of distinct knots, so we can derive a relationship between their knot determinant and their structure.

A torus knot is one that can be drawn on a torus without self-intersection. A torus knot can be expressed as T(n, k), where n is the number of times a knot wraps the meridional direction, while k is the number of times it wraps around the longitudinal direction. Equivalently, if the surface of the torus is cut and unfolded into a rectangle, the knot can be viewed as travelling $n$units horizontally and $k$units vertically before reconnecting to its starting point (DeMason & Hager, 2025).

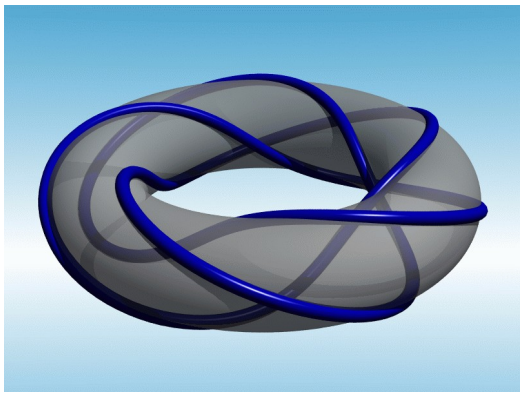

Figure 9. A Torus Knot around a Torus
(Duke University, 1998)

Breiland et al. (2009) show that for n and k relatively prime,

$$det\left(T\left(n,k\right)\right) = \begin{cases} k, if\ n\ is\ even \wedge k\ is\ odd, \\ n, if\ n\ is\ odd \wedge k\ is\ even, \\ 1, if\ n\ is\ odd \wedge k\ is\ odd. \end{cases}$$

Case 1. If *n* is even and *k* is odd.

We can test this using T(2,3), the trefoil.

Its coloring matrix is:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

And its minor is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Therefore, it has determinant 3, which is equal to *k*.

Case 2. If *n* is odd and *k* is even.

We can test this using the T(3,2) knot. This is the same knot as T(2,3). So, the coloring matrix is identical up to relabeling, and its determinant is again 3. This verifies the second parity case.

Case 3. If *n* is odd and *k* is odd.

This case is verified using T(3,5). From Knot Atlas (2023), the determinant of this knot is 1, which agrees with the determinant formula.

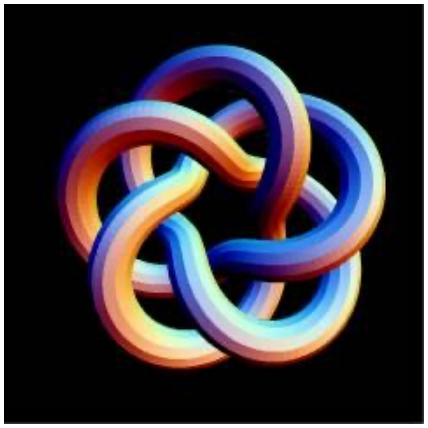

Figure 10. A T(3,5) knot

(The Knot Atlas, 2023)

This pattern allows the determinant of any torus knot to be determined without a coloring matrix. If we use the multiplicative property established in section 4, we can calculate the determinant of a composite knot formed from two torus knots immediately as well.

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